DEGENERATION OF PROJECTIVE SPACE WITH EXTENDABLE HYPERPLANE CLASS

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Abstract. The goal of this note is to give an exposition on the following fact: for a degeneration of $\mathbb{P}^n$ over a DVR such that the hyperplane divisor extends to a Cartier divisor, if the special fiber is geometrically integral then it must also be $\mathbb{P}^n$.

We start with a characterization of $\mathbb{P}^n$ which appeared in [Fuj90, Thm 1.1, pg. 22]. The result is slightly generalized as we drop the assumption that X is Cohen Macaulay.

Theorem 1. *Let X be a proper geometrically integral variety of dimension n over a field $\mathbb{k}$, with an ample line bundle L such that $L^n = 1$ and $h^0(X, L) \geqslant n + 1$. Then we have an isomorphism $X \xrightarrow{|L|} \mathbb{P}^n_{\mathbb{k}}$.*

Proof. Consider the algebraic closure $\bar{\mathbb{k}}$. By flat base change we get that

$$H^0(X, L) \otimes_{\mathbb{k}} \bar{\mathbb{k}} \simeq H^0(X_{\bar{\mathbb{k}}}, L_{\bar{\mathbb{k}}})$$

which also implies that $L^n_{\bar{\mathbb{k}}} = 1$ since intersection number can be defined via Euler characteristic. Suppose that we have an isomorphism

$$X_{\bar{\mathbb{k}}} \xrightarrow{|L_{\bar{\mathbb{k}}}|} \mathbb{P}^n_{\bar{\mathbb{k}}}$$

Since $\mathbb{k} \to \bar{\mathbb{k}}$ is faithfully flat, this is a fpqc base change of $X \xrightarrow{|L|} \mathbb{P}^n_{\mathbb{k}}$. Isomorphism can be checked fpqc-local (see [Stacks, Tag 02L4]), so we get an isomorphism $X \xrightarrow{|L|} \mathbb{P}^n_{\mathbb{k}}$. Thus it suffices to prove the theorem in the case $\mathbb{k} = \bar{\mathbb{k}}$.

We will prove the characterization using induction, with the base case $\dim X = 1$ trivial. Consider the normalization $\tilde{X} \xrightarrow{\upsilon} X$. This is a birational finite map, so we get

$$(\upsilon^* L)^n = 1, \quad \upsilon^* L \text{ is ample}$$

and since it's dominant (with X integral), we get an injection $H^0(X, L) \hookrightarrow H^0(\tilde{X}, \upsilon^* L)$. Thus $h^0(\tilde{X}, \upsilon^* L) \geqslant n + 1$.

Consider $x \in \tilde{X}$. There exists a nonzero $s \in H^0\left(\tilde{X}, \upsilon^* L\right)$ such that $s(x) = 0$ (possible since $h^0\left(\tilde{X}, \upsilon^* L\right) \geqslant 2$), and the corresponding Cartier divisor $D = V(s) \ni x$. Write

$$D_{\text{red}} = \sum D_j, \quad D = \sum m_j D_j \quad m_j \in \mathbb{Z}_+$$

then

$$1 = (\upsilon^* L)^{n-1} \cdot D = \sum m_j ((\upsilon^* L)^{n-1} \cdot D_j) \geqslant \sum m_j$$

since $\upsilon^* L$ is ample. Thus D is irreducible, and generically reduced. Furthermore, $\tilde{X}$ is normal hence it has property (S_2), which implies that D has (S_1). D then satisfies both (S_1)

Date: August 1, 2026.

and (R_0), so D must be integral (see [Stacks, Tag 031R]). On the other hand, we have an exact sequence:

$$0 \to H^0\left(\tilde{X}, \mathcal{O}_{\tilde{X}}\right) \simeq \Bbbk \to H^0\left(\tilde{X}, \upsilon^* L\right) \to H^0(D, \upsilon^* L|_D) \to H^1\left(\tilde{X}, \mathcal{O}_{\tilde{X}}\right)$$

which implies that $h^0(D, \upsilon^* L|_D) \geqslant h^0\left(\tilde{X}, \upsilon^* L\right) - 1 = n$. We also have $(\upsilon^* L|_D)^{n-1} = (\upsilon^* L)^{n-1} \cdot D = 1$. Clearly $\upsilon^* L|_D$ is still ample, so we can use the inductive hypothesis to get an isomorphism

$$D \xrightarrow{|\upsilon^* L|_D|} \mathbb{P}^{n-1}_{\Bbbk}$$

Now, $h^0(D, \upsilon^* L|_D) = n$, so in the previous exact sequence we actually get a surjection

$$H^0\left(\tilde{X}, \upsilon^* L\right) \to H^0(D, \upsilon^* L|_D) \to 0$$

with $h^0\left(\tilde{X}, \upsilon^* L\right) = n + 1$. Looking at $x \in D \simeq \mathbb{P}^{n-1}_{\Bbbk}$, we can pick a section not vanishing at x and lift it to $H^0\left(\tilde{X}, \upsilon^* L\right)$. This is true for all $x \in D$, so $\upsilon^* L$ is base-point-free, giving a morphism

$$\phi : \tilde{X} \xrightarrow{|\upsilon^* L|} \mathbb{P}^n$$

with $\upsilon^* L = \phi^* \mathcal{O}_{\mathbb{P}^n}(1)$. If some fiber $\tilde{X}_p$ is positive dimensional, then $\phi^* \mathcal{O}_{\mathbb{P}^n}(1)|_{\tilde{X}_p} = \mathcal{O}_{\tilde{X}_p}$ which contradicts ampleness of $\upsilon^* L$. Thus ϕ must be quasi-finite; it's also proper, so ϕ is a finite map. $(\upsilon^* L)^n = 1$ then gives that the degree of the map must be 1. By Zariski's Main Theorem, ϕ is an isomorphism.

On the other hand, since $h^0\left(\tilde{X}, \upsilon^* L\right) = n + 1$, we actually have an isomorphism

$$\upsilon^* : H^0(X, L) \xrightarrow{\simeq} H^0(\tilde{X}, \upsilon^* L)$$

so we have a factorization

$$\begin{tikzcd} \tilde{X} \arrow[rr, "|\upsilon^* L|"] \arrow[dr, "\upsilon"'] & & \mathbb{P}^n \\ & X \arrow[ur, "|L|"'] & \end{tikzcd}$$

and by the same argument using Zariski's Main Theorem, we get that $X \xrightarrow{|L|} \mathbb{P}^n$ is an isomorphism. ∎

The next result adapts an argument of Hironaka (see [Hir68, Lemma 1.7]) in the case where the special fiber is smooth.

Theorem 2. *Let $\pi : X \to \operatorname{Spec} R$ be a proper flat family over a DVR $(R, \mathfrak{m}, \Bbbk)$ with $K = \operatorname{Frac}(R)$. Suppose that the generic fiber $X_K \simeq \mathbb{P}^n_K$, the special fiber $X_{\Bbbk}$ is geometrically integral, and the effective hyperplane divisor $H_K \in H^0(X_K, \mathcal{O}_{X_K}(1))$ extends to a Cartier divisor H on X. Then $X_{\Bbbk} \simeq \mathbb{P}^n_{\Bbbk}$.*

Proof. If we can show that H is $\pi-$ample, then local constancy of Hilbert polynomial (by flatness of π), along with upper-semicontinuity of cohomology gives that $H_{\Bbbk} = H|_{X_{\Bbbk}}$ satisfies all the conditions in theorem 1, and we get the desired isomorphism. H is ample on the generic fiber by definition, so it remains to show that $H_{\Bbbk}$ is ample on $X_{\Bbbk}$. We will make use of Nakai-Moishezon-Kleiman criterion by showing that for every positive dimensional integral subscheme $Z \subset X_{\Bbbk}$, $\left(H_{\Bbbk}^{\dim Z} \cdot Z\right) > 0$. We will induct on $\dim Z$.

Claim 3. *It suffices to find an effective Cartier divisor $H'_{\Bbbk} \in |mH_{\Bbbk}|$ with $m > 0$ such that $H'_{\Bbbk}$ does not contain Z.*

Proof of claim. If Z is a curve, then $H'_{\Bbbk} \cap Z$ in points, and $m(H_{\Bbbk} \cdot Z) \geqslant \#\{H'_{\Bbbk} \cap Z\} > 0$. For the inductive step, we have $H'_{\Bbbk} \cap Z = \sum a_j Z_j$ with $a_j \in \mathbb{Z}_+$, $\dim Z_j = \dim Z - 1$, and

$$m(H_{\Bbbk}^{\dim Z} \cdot Z) = H_{\Bbbk}^{\dim Z - 1}(H'_{\Bbbk} \cdot Z) = \sum a_j (H_{\Bbbk}^{\dim Z-1} \cdot Z_j) > 0$$

by the inductive hypothesis. ∎

We will now construct such $H'_{\Bbbk}$. Consider $z \in Z$. Suppose that Z is locally defined by $\mathfrak{p}_Z \subset \mathfrak{m}_{X,z} \subset \mathcal{O}_{X,z}$. Since Z is positive dimensional, there exists some $g \in \mathfrak{m}_{X,z} - \mathfrak{p}_Z$. Take a minimal prime $\mathfrak{p}$ lying over (g). Then the closure $H' = \overline{\{\mathfrak{p}\}}$ is a Weil divisor in X. Define $H'_{\Bbbk} = H'|_{X_{\Bbbk}}$; it remains to show that $H'_{\Bbbk} \simeq mH_{\Bbbk}$ as Cartier divisors for $m > 0$. We have an exact sequence

$$\mathbb{Z}[X_{\Bbbk}] \to \mathrm{Cl}(X) \to \mathrm{Cl}(X_K) \to 0$$

and since $X_{\Bbbk}$ is a principal Weil divisor, we get an isomorphism $\mathrm{Cl}(X) \xrightarrow{\simeq} \mathrm{Cl}(X_K)$ by restriction. Notice that H' intersects the generic fiber X_K properly, so $H'_K \sim mH_K$, for some $m > 0$, as both Cartier and Weil divisors. It follows that $H' \sim mH$, with $m > 0$, as Weil divisors, i.e. there is some rational function $f \in K(X)^\times$ such that

$$H' = mH + \mathrm{div}(f)$$

and in particular H' is locally principal. Since X is normal (see [Liu06, Lemma 1.18, pg. 119]), we get that H' is a Cartier divisor by [Har77, Remark 6.11.2]. As a result, $H'_{\Bbbk} \sim mH_{\Bbbk}$, for $m > 0$, as Cartier divisors and we are done. ∎

Corollary 4. *Let $\pi : X \to \operatorname{Spec} R$ be a proper flat family over a DVR $(R, \mathfrak{m}, \Bbbk)$ with $K = \mathrm{Frac}(R)$. If $X_K \simeq \mathbb{P}^n_K$, and the special fiber $X_{\Bbbk}$ is smooth, then $X_{\Bbbk} \simeq \mathbb{P}^n_{\Bbbk}$.*

Proof. Here we have all smooth fibers over a regular base, so X must be regular. Let H be the closure of H_K in X, then H is a Weil divisor hence a Cartier divisor (since X is regular). ∎

References

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