1 Introduction

The goal is to prove the following theorem:

Theorem 1.1. *Let S be a smooth algebraic variety over $\mathbb{C}$, and $o \in S$. Let $N \in \mathbb{Z}_+$ be fixed. Then there are only finitely isomorphism classes of representations*

$$\rho : \pi_1(S, o) \to \mathrm{GL}_N(\mathbb{Q})$$

coming from geometry. Here we say that a representation comes from geometry if it's isomorphic to a subquotient of a monodromy representation attached to a smooth and projective map $f : X \to S$.

This follows from a stronger statement:

Theorem 1.2. *Let S be a connected complex manifold, and $o \in S$ such that $\pi_1(S, o)$ is finitely generated. Let $N \in \mathbb{Z}_+$, then*

1. *There exist only finitely many isomorphism classes of $\mathbb{Q}-$local systems of rank N on S underlying a polarizable integral variation of Hodge structures, up to semi-simplification.*
2. *If S is compactifiable (i.e., there exists a compact complex manifold $\overline{S}$ such that $S = \overline{S} - Z$ where Z is a closed analytic subset). Then there exist only finitely many isomorphism classes of $\mathbb{Q}-$local systems of rank N which are subquotients of local systems underlying polarizable integral variation of Hodge structures.*

Note 1.1. Notice that the first part is only up to semi-simplification, so that's why in part 2 we need a stronger condition.

Proof of theorem 1.1. By Nagata compactification theorem, there is a proper variety $\overline{S}$ containing S. Then by Hironaka's resolution of singularities we can assume that $\overline{S}$ is smooth hence a manifold (we only need to blow up singular points, which are in $\overline{S} - S$).

If $\mathbf{V}$ is a $\mathbb{Q}-$local system coming from geometry, then $\mathbf{V}$ is a subquotient of $\mathbf{H} = \mathrm{R}^n f_* \mathbb{Q}_X$ for some $f : X \to S$. $\mathbf{H}$ underlies a polarizable integral variation of Hodge structures, hence by part 2 of theorem 1.2, there are only finitely many such local systems. □

Note 1.2. Another point of note here is that an algebraic variety S has a finite CW-complex structure, hence the fundamental group is finitely presented. For the former claim, see here. The idea is that a pair (semi-algebraic set, closed subset) in $\mathbb{R}^n$ can be triangulated, hence quasi-projective varieties have finite CW-complex structures. S can be compactified (by Nagata) to $\overline{S}$, and by Chow's lemma $\overline{S}$ is birational (i.e., can be blown up to) a projective variety $\widetilde{S}$. Then $(\widetilde{S}, \widetilde{S} - S)$ can be triangulated, thus S has a finite CW-complex structure.

For the latter claim of finitely presented fundamental group, any map $\gamma : \mathbb{S}^1 \to S$ is homotopic to a cellular map. Any two cellular maps are homotopic through a cellular homotopy, i.e., a homotopy that is cellular. Hence we only need to care up to a cellular map $\mathbb{S}^1 \times I \to S$, i.e., only cares up to the 2-skeleton $S^{(2)}$. In fact, $\pi_1(S^{(1)}) \to \pi_1(S^{(2)})$ is surjective (since we haven't identified the cellular maps that are homotopic), and $\pi_1(S^{(2)}) \simeq \pi_1(S)$. Now, $S^{(1)}$ is just a finite graph, hence $\pi_1(S)$ is finitely generated. For finitely presented we need to work a bit more to figure out the kernel.

The compactifiable condition comes from Schmid's theorem of the fixed part:

Theorem 1.3. *Let S be a compactifiable complex manifold and $\mathbf{V}$ is a polarized complex variation of Hodge structures. Then any global flat section of $\mathbf{V}$ (i.e., a section of the underlying local system) has flat components.*

Corollary 1.4. *Let $\mathbf{V}$ be a local system on (S, o) underlying a polarizable variation of $\mathbb{Q}$-Hodge structure. Then $H^0(S, \mathbf{V})$, which can be identified with*

$$\mathbf{V}_o^{\pi_1(S,o)} = \left\{v \in \mathbf{V}_s \middle| \gamma \cdot v = v \ \forall \ \gamma \in \pi_1(S, o)\right\}$$

has a natural $\mathbb{Q}-$Hodge structure such that the restriction map $H^0(S, \mathbf{V}) \to \mathbf{V}_o$ is a morphism of Hodge structure. Furthermore, the image is $\mathbf{V}_o^{\pi_1(S,o)}$.

Note 1.3. Sanity check: it should be the case then that the restriction map $H^0(S, \mathbf{V}) \to \mathbf{V}_o$ is injective. Consider $s \in H^0(S, \mathbf{V})$ and take $\{U_i\}$ to be a trivialization of S. Suppose s is 0 after restricted to $\mathbf{V}_o$ then $s|_{U_i} = 0$ for some $U_i \ni o$. Since S is connected there must be some other U_j intersecting U_i, hence $s|_{U_i \cup U_j} = 0$. Due to connectedness again, we must be able to find a different $U_k \neq U_i, U_j$ intersecting $U_i \cup U_j$, and repeating this process we get that $s = 0$ to begin with.

Restriction being injective actually true for any coherent torsion-free sheaf on an integral scheme. For a functorial identification of $H^0(S, \mathbf{V})$, look at lemma 4.17 in Voisin's vol 2. The main ingredients are that a morphism of local systems $\phi : \mathbf{V} \to \mathbf{W}$ is just a map on fibers $\phi_o : \mathbf{V}_o \to \mathbf{W}_o$ which is $\pi_1(S, o)-$equivariant, and that

$$H^0(S, \mathbf{V}) = \mathrm{Hom}_{\mathbb{Z}_S}(\mathbb{Z}_S, \mathbf{V})$$

which follows from the fact that a local system of abelian groups is just a locally constant sheaf of $\mathbb{Z}_S-$modules (and then recall $H^0(X, \mathscr{F}) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F})$ for sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules).

Note 1.4. Another version (that Ben likes to use) states that the sub-local-system of $\mathbf{V}$ of $\pi_1(S, o)-$invariant vectors on each stalk is a sub-VHS. This is just the constant sheaf with stalk $H^0(S, \mathbf{V})$.

In order to prove theorem 1.2, we will need the following theorems:

Theorem 1.5. *Let (S, o) be as in theorem 1.2. Consider the equivalence condition: let $\rho_1, \rho_2 : \pi_1(S, o) \to \mathrm{GL}_N(\mathbb{C})$, then $\rho_1 \sim \rho_2$ if $\mathrm{Tr}(\rho_1(\gamma)) = \mathrm{Tr}(\rho_2(\gamma))$ for all $\gamma \in \pi_1(S, o)$. Then the set*

$$\left\{\text{local system } \mathbf{H} \text{ of rank } N \text{ underlying integral polarizable VHS}\right\}\big/\sim$$

is finite.

Theorem 1.6. *Let (S, o) be a compactifiable connected complex manifold, and let $\mathbf{H}$ be a $\mathbb{C}-$local system underlying an integral polarizable variation of Hodge structures. Then $\mathbf{H}$ is semisimple, i.e.,*

$$\mathbf{H} = \bigoplus W_i \otimes \mathbf{L}_i$$

where $\mathbf{L_i}$'s are pairwise non-isomorphic irreducible local systems, and W_i's are complex vector spaces. Furthermore, we can put Hodge structures on W_i, and VHS on $\mathbf{L}_i$ to make this an equality of complex polarized VHS.

Theorem 1.7. *Now let $\mathbf{V}$ be a direct summand of $\mathbf{H}$. Then $\mathbf{V}$ admits a polarized VHS.*

Proof of theorem 1.2. This follows from a more general result: Let A be a $\Bbbk-$algebra with $\mathrm{char}(\Bbbk) = 0$, and M, N be semisimple $A-$modules which are finite dimensional over $\Bbbk$. Each $a \in A$ defines, by multiplication, an element in $a_M \in \mathrm{End}_{\Bbbk}(M)$ (and $a_N \in \mathrm{End}_{\Bbbk}(N)$). If $\mathrm{Tr}(a_M) = \mathrm{Tr}(a_N)$ for all $a \in A$ then $M \simeq_A N$. See here.

The idea is that this is true for A finite-dimensional over $\Bbbk$ (equivalently, A artinian, see Lam's Noncommutative rings, theorem 7.19), and to reduce to that case we take B to be the image of

$$A \to \mathrm{End}(M \oplus N), \quad a \mapsto (a_M, a_N)$$

then B is Artinian and $M \simeq_B N$ which implies $M \simeq_A N$ (notice $a_M(m) = (a_M, a_N) \cdot m = (a_M, a_N) \cdot n = a_N(n)$). In our case, let $A = \mathbb{Q}[\pi_1(S, o)]$ then the result follows.

For the second part, let $\mathbf{V}$ be a subquotient of $\mathbf{H}$ which underlies a polarized $\mathbb{Z}-$VHS. Then by theorem 1.6, $\mathbf{V}$ is a direct summand, hence underlies a polarized VHS. By (a stronger version which doesn't require integrality) theorem 1.6, $\mathbf{V}$ is semisimple, hence by the first part we get the desired result. $\square$

2 Proofs

In order to prove theorem 1.5, we will first show that for a fixed $\gamma \in \pi_1(S, o)$ and $N \in \mathbb{Z}_+$, there is a bound for $\mathrm{Tr}(\rho(\gamma))$ for all local systems underlying polarized VHS of rank N.

Proposition 2.1. *Let (S, o) be a connected complex manifold, $\gamma \in \pi_1(S, o)$ and $N \in \mathbb{Z}_+$. Then there exists $C > 0$ such that $|\mathrm{Tr}(\rho(\gamma))| < C$ for all $\rho : \pi_1(S, o) \to \mathrm{GL}(\mathbf{H}_o)$ where $\mathbf{H}$ is a polarized VHS of rank N.*

Proof. Consider $\mathbf{H}$ a polarized VHS of rank N. We have a period map $p : S \to \Gamma\backslash D$ where $D = G/K = \mathrm{Aut}(\mathbf{H}_o, q) \cap \mathrm{SL}(\mathbf{H}_o)$ and K is the subgroup fixing the flag corresponding to o. The main thing is that K is a compact subgroup (see CMSP proposition 4.4.4). This lifts to a $\pi_1(S, o)$-equivariant map on universal cover

$$P : \widetilde{S} \to D$$

where $P(\gamma \cdot o) = \rho(\gamma)(P(o))$. We will need a lemma (see CMSP corollary 13.7.2)

Lemma 2.2. *There exists a $G-$invariant metric d_D on D such that every horizonal holomorphic map $f : \Delta \to D$ is distance decreasing, i.e.,*

$$d_D(f(x), f(y)) \leq d(x, y) \quad \forall\ x, y \in \Delta$$

where d is the Poincare metric on the unit disk.

Proof of lemma 2.2. We have the trace form on G, and combining with the Weil operator this gives a $G-$invariant metric on D. The holomorphic sectional curvature is negative and bounded away from 0 (CMSP, theorem 13.6.3). Hence we can normalize the metric to something with sectional curvature ≤ -1. Then by Schwarz-Ahlfors-Pick's theorem every holomorphic map from the unit disk is distance decreasing. $\square$

We can put a Kobayashi metric d_S on $\widetilde{S}$ such that

$$d_D(P(o), \rho(\gamma)(P(o))) \leq d_S(o, \gamma \cdot o)$$

and the claim is that $d_S(x, y)$ is finite for all $x, y \in \widetilde{S}$ since it is connected (this probably has to do with the construction of the Kobayashi metric, see Roydan's Remarks on the Kobayashi metric; essentially the more paths you use to connect 2 points, the lower the sum of distances drops). Let this bound be M.

Next, (D, d_D) is Riemannian homogeneous (isometries act transitively) since d_D is G-invariant, hence (D, d_D) is complete. It follows that the closed ball are compact (in a geodesically complete Riemannian manifold, a subset is compact iff it's closed and bounded). Thus

$$\left\{\rho(\gamma) \in G \middle| d_D(P(o), \rho(\gamma)(P(o))) \le M\right\}$$

is compact in D. Notice that this set only depends on (D, d_D) which only depends on the Hodge numbers $h^{p,q}$ (see CMSP proposition 4.4.4, even the polarization goes away and we are left with just symplectic and orthogonal groups).

Next observe that $D = G/K$ and K is compact with d_D being $G-$invariant, so $\rho(\gamma)$ is in a bounded set in G. It follows that its entries must be bounded, hence $\mathrm{Tr}(\rho(\gamma))$ is bounded, and this bound only depends on the Hodge numbers.

To conclude the proof, we need to show that there are only finitely many possibilities for $h^{p,q}$. We will do this by induction on the rank N, with the case $N = 0$ being trivial. Now let w be the weight of $\mathbf{H}$. Suppose there exists $i < p < j$ such that

$$h^{i,w-i} \neq 0, \quad h^{p,w-p} = 0, \quad h^{j,w-j} \neq 0$$

then $F^{p+1}\mathbf{H}$ satisfies Griffiths transversality (the only piece we have to worry about is ∇F^{p+1} which might end up in F^p which is not contained in F^{p+1} however $h^{p,w-p} = 0$ so $F^p = F^{p+1}$). So F^{p+1} is a sub-VHS, and by theorem 1.6, we can decompose $\mathbf{H}$ as a direct sum of 2 sub-VHS of strictly smaller rank. Hence induction takes care of this case.

For the case where $\left\{p \middle| h^{p,w-p}\right\} \neq 0$ is an integer interval, we can twist by the Tate module and assume that $w \le N$. In this case we also have that $h^{p,q}$ can only take finitely many values. □

Proposition 2.3. *Let Γ be a finitely generated group and $N \in \mathbb{Z}_+$. There exists a finite subset $F \subset \Gamma$ such that if the traces of $\rho_1, \rho_2 : \Gamma \to \mathrm{GL}_N(\mathbb{C})$ agree on all $\gamma \in F$, then they agree on all $\gamma \in \Gamma$.*

Proof. The (morally correct) idea here is that $\mathrm{Hom}(\Gamma, \mathrm{GL}_N(\mathbb{C}))$ (the $\mathbb{C}-$points of the representation scheme) is an affine variety with coordinate ring A. Procesi showed that $A^{\mathrm{GL}_N(\mathbb{C})}$ is generated by

$$\left\{\mathrm{Tr}(\gamma) : \rho \mapsto \mathrm{Tr}(\rho(\gamma)) \middle| \gamma \in \Gamma\right\}$$

and since Γ is finitely generated, so is A thus we can pick finitely many $\gamma \in \Gamma$ to generate the invariant set. □

Proof of theorem 1.5. Take a set $F \subset \pi_1(S, o)$ as in the previous proposition. Any $\rho : \pi_1(S, o) \to \mathrm{GL}(\mathbf{H}_o)$ underlying an integral polarizble VHS must factor through $\mathrm{GL}_N(\mathbb{Z})$, i.e., $\rho(\gamma)$ is an integer matrix for all $\gamma \in \pi_1(S, o)$.

By proposition 2.1, for a fixed $\gamma \in \pi_1(S, o)$, $\mathrm{Tr}(\rho(\gamma))$ can only take finitely many values as the local system $\mathbf{H}$ varies (bounded + integer value implies finite possibilities). Then on F, the traces can only take finitely many values as well (since F is finite), and then by the previous proposition we get the desired conclusion. □

Proof of theorem 1.7. We have a local system $\mathbf{End}(\mathbf{H})$ whose stalk at o is $\mathrm{End}(\mathbf{H}_o)$. By theorem of the fixed part, the global sections $\mathrm{End}(\mathbf{H})$ has a Hodge structure compatible with restriction. Now, by theorem 1.6,

$$\mathbf{H} = \bigoplus W_i \otimes \mathbf{L}_i$$

so by Schur's lemma,

$$\mathrm{End}(\mathbf{H}) = \prod \mathrm{End}(W_i)$$

One can show that any grading of $\prod \mathrm{End}(W_i)$, compatible with restriction, has to come from gradings of W_i. Fix such gradings, and assume $\mathbf{V} = \mathbf{L}_j$. Then a homogenous (contained in a graded piece) line $L \subset W_j$ defines a projection of degree 0 in $\mathrm{End}(W_i)$ with image in L. This, in turn, induces a projection $\mathbf{H} \to L \otimes \mathbf{L}_j \simeq \mathbf{V}$. $\square$