PERVERSE EULER CHARACTERISTICS OF HERMITIAN LOCALLY SYMMETRIC SPACES

MERT AKDENIZLI AND QUOC-ANH TRAN

Abstract. We prove that finite-volume locally Hermitian symmetric spaces of noncompact type have nonnegative perverse Euler characteristics. To show this, we obtain a nefness result for the logarithmic cotangent bundle of a smooth toroidal compactification. Combining this with a positivity criterion for Euler characteristics of perverse sheaves, we deduce the nonnegativity result. We further prove that the inequality is strict for perverse sheaves with full support. As applications, we get nonnegativity results for perverse Euler characteristics on various moduli spaces.

Contents

1. Introduction and main results

Let $\mathcal{D}$ be a Hermitian symmetric domain of noncompact type and let

$$G_{\mathcal{D}} = \mathrm{Aut}(\mathcal{D})^{\circ}.$$

For a lattice $\Gamma \subset G_{\mathcal{D}}$, the quotient $\Gamma\backslash\mathcal{D}$ is a finite-volume locally Hermitian symmetric orbifold. We study the nonnegativity of Euler characteristics of perverse sheaves on the Deligne–Mumford stack attached to this orbifold. Let

$$\mathcal{X}_{\Gamma} := [\mathcal{D}/\Gamma]$$

denote the associated analytic Deligne–Mumford quotient stack. Its coarse space is the locally symmetric quotient $\Gamma\backslash\mathcal{D}$. All constructible complexes and perverse sheaves on $\mathcal{X}_{\Gamma}$ are understood in the analytic topology. We write

$$\chi^{\mathrm{orb}}(\mathcal{X}_\Gamma, K)$$

for the Euler–Satake characteristic of a constructible complex $K \in D^b_c(\mathcal{X}_\Gamma, \mathbb{Q})$. Its definition and its behavior under finite étale covers are recalled in Section 2.1. If Γ is torsion-free, then $\mathcal{X}_\Gamma$ is represented by the complex manifold $X_\Gamma = \Gamma\backslash\mathcal{D}$, which carries its natural smooth quasi-projective algebraic structure. In this case χ^{orb} is the ordinary Euler characteristic.

Our main result is the following:

Theorem 1.1 (Main theorem)**.** *Let $\mathcal{D}$ be a Hermitian symmetric domain of noncompact type and let $\Gamma \subset \mathrm{Aut}(\mathcal{D})^\circ$ be a lattice. Then*

$$\chi^{\mathrm{orb}}(\mathcal{X}_\Gamma, P) \geq 0$$

for every $P \in \mathrm{Perv}(\mathcal{X}_\Gamma, \mathbb{Q})$. If Γ is torsion-free, so that $X_\Gamma = \Gamma\backslash\mathcal{D}$ is a smooth quasi-projective variety, then

$$\chi(X_\Gamma, P) \geq 0 \qquad \text{for every } P \in \mathrm{Perv}(X_\Gamma, \mathbb{Q}).$$

If X is a smooth irreducible complex variety of dimension n and $P \in \mathrm{Perv}(X, \mathbb{Q})$ has full support, then there is a dense Zariski-open subset $U \subset X$ such that $P|_U \simeq L[n]$ for a local system L on U. The rank of L is called the generic rank of P and is denoted by $r(P)$. If $\mathrm{Supp}(P) \neq X$, we set $r(P) = 0$.

For a perverse sheaf on $\mathcal{X}_\Gamma$, we define its generic rank to be the generic rank of its pullback to a neat finite étale cover $Y \to \mathcal{X}_\Gamma$. This is independent of the chosen cover because any two finite tower admits a common one finite etale pullback does not change the rank of a local system.

Following [2] we also improve the result to strict inequality for perverse sheaves with full support.

Proposition 1.2. *With the notation of Theorem 1.1, let $n = \dim_{\mathbb{C}} \mathcal{D}$. Then for every $P \in \mathrm{Perv}(\mathcal{X}_\Gamma, \mathbb{Q})$,*

$$\chi^{\mathrm{orb}}(\mathcal{X}_\Gamma, P) \geq r(P)(-1)^n \chi^{\mathrm{orb}}(\mathcal{X}_\Gamma).$$

In particular for a perverse sheaf P which has full support we have

$$\chi^{\mathrm{orb}}(\mathcal{X}_\Gamma, P) > 0.$$

The stack formulation is needed when the lattice has torsion. In particular, the first assertion of Theorem 1.1 concerns the Deligne-Mumford quotient $\mathcal{X}_\Gamma$, rather than the possibly singular coarse quotient $\Gamma\backslash\mathcal{D}$.

Theorem 1.1 gives a finite-level statement for Shimura varieties. Let $(G, \mathcal{X})$ be a Shimura datum and let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Choose a normal neat compact open subgroup $K' \triangleleft K$ and define the finite-level Deligne-Mumford Shimura stack by

$$\mathrm{Sh}^{\mathrm{DM}}_K(G, \mathcal{X}) := [\mathrm{Sh}_{K'}(G, \mathcal{X})/(K/K')]. \tag{1}$$

This is an algebraic Deligne-Mumford stack. Its coarse moduli space is the usual finite-level Shimura variety $\mathrm{Sh}_K(G, \mathcal{X})$.

Corollary 1.3. *Let $(G, \mathcal{X})$ be a Shimura datum and let $K \subset G(\mathbb{A}_f)$ be an arbitrary compact open subgroup. Then*

$$\chi^{\mathrm{orb}}(\mathrm{Sh}^{\mathrm{DM}}_K(G, \mathcal{X})_{\mathbb{C}}, P) \geq 0$$

for every

$$P \in \mathrm{Perv}(\mathrm{Sh}^{\mathrm{DM}}_K(G, \mathcal{X})_{\mathbb{C}}, \mathbb{Q}).$$

Strict inequality holds whenever P has full support. When K is neat, the stack $\mathrm{Sh}^{\mathrm{DM}}_K(G, \mathcal{X})_{\mathbb{C}}$ is represented by the usual smooth finite-level Shimura variety, and χ^{orb} becomes the ordinary Euler characteristic.

The geometric input needed is the following nefness theorem for toroidal compactifications of Shimura varieties.

Theorem 1.4. *Let X be a connected component of a Shimura variety at neat finite level, and let*

$$j\colon X \hookrightarrow \overline{X}$$

be a smooth projective toroidal compactification whose boundary $D = \overline{X} \setminus X$ is a simple normal-crossings divisor. Then

$$\Omega^1_{\overline{X}}(\log D)$$

is nef.

For complex-ball quotients, the corresponding statement is due to Cadorel.

Theorem 1.5 (Cadorel)**.** *Let $\Delta \subset \mathrm{Aut}(\mathbb{B}^n)$ be a finite-volume lattice whose parabolic elements are unipotent, and let*

$$X = \Delta \backslash \mathbb{B}^n.$$

Let

$$X \hookrightarrow \overline{X}$$

be its smooth projective toroidal compactification and let $D = \overline{X} \setminus X$ be the boundary. Then

$$\Omega^1_{\overline{X}}(\log D) \quad \textit{is nef.}$$

Proof. This is [8, Theorem 3]. □

Section 2 will reduce the main problem to finite level Shimura variety and a ball quotient case. After obtaining the nefness results in each case the product nefness will follow. The logarithmic characteristic-cycle criterion of Arapura–Patel [3] and finite étale invariance will give the desired result on $\mathcal{X}_\Gamma$.

Nefness results on finite level Shimura setting is proved using the adjoint variation of Hodge structure and positivity results on its canonical extension. Together with Theorem 1.5, it supplies the two geometric inputs in the preceding reduction. The final part of the paper applies the resulting Euler-characteristic inequalities to moduli spaces admitting locally symmetric period descriptions.

2. Preliminaries and the reduction theorem

2.1. **The PEC property and the logarithmic criterion**

Let U be a complex algebraic variety and let $K \in D^b_c(U, \mathbb{Q})$. We have

$$\chi(U, K) = \sum_{j \in \mathbb{Z}} (-1)^j \dim_{\mathbb{Q}} \mathbb{H}^j(U, K).$$

We say that U has the perverse Euler-characteristic property, or briefly the PEC property, if

$$\chi(U, P) \geq 0 \qquad \text{for every } P \in \mathrm{Perv}(U, \mathbb{Q}). \tag{2}$$

The following logarithmic criterion will be used throughout the paper.

Theorem 2.1 (Arapura–Patel lemma 2.2 [3])**.** *Let*

$$U = \overline{U} \setminus D,$$

where $\overline{U}$ is a smooth projective complex variety and D is a simple normal-crossings divisor. If

$$\Omega^1_{\overline{U}}(\log D)$$

is nef, then U has the PEC property; equivalently,

$$\chi(U, P) \geq 0 \qquad \textit{for every } P \in \mathrm{Perv}(U, \mathbb{Q}).$$

Let $\mathcal{Y}$ be a finite-type complex Deligne–Mumford stack. For a complex Deligne–Mumford stack, all constructible complexes and perverse sheaves below are taken on the underlying analytic stack with $\mathbb{Q}$-coefficients. Following [3, §2.6], we write

$$\chi^{\mathrm{orb}}(\mathcal{Y}, K) \in \mathbb{Q}$$

for the Euler–Satake characteristic of $K \in D^b_c(\mathcal{Y}, \mathbb{Q})$. We say that $\mathcal{Y}$ has the PEC property if

$$\chi^{\mathrm{orb}}(\mathcal{Y}, P) \geq 0 \qquad \text{for every } P \in \mathrm{Perv}(\mathcal{Y}, \mathbb{Q}). \tag{3}$$

When $\mathcal{Y}$ is a variety, its Euler–Satake characteristic is the ordinary Euler characteristic [3, Theorem 2.5(i)]. More generally, if

$$\mathcal{Y} = [Y/F]$$

for a finite group F acting on a variety Y, and $q\colon Y \to \mathcal{Y}$ is the quotient atlas, then

$$\chi^{\mathrm{orb}}(\mathcal{Y}, K) = \frac{1}{|F|}\chi(Y, q^*K). \tag{4}$$

This is the finite étale degree formula of [3, Theorem 2.5(iii)].

We use the perverse t-structure on Deligne–Mumford stacks in its smooth-local form; see [18, §4]. Thus, if $u\colon U \to \mathcal{Y}$ is a smooth atlas of pure relative dimension e, then P is perverse on $\mathcal{Y}$ if and only if $u^*P[e]$ is perverse on U. In particular, pullback by a representable étale morphism is t-exact.

All locally symmetric Deligne–Mumford stacks considered below admit representable finite étale atlases by varieties, by the quotient presentation above. The substacks occurring in the applications inherit such atlases by base change. We record finite étale invariance in this setting.

Proposition 2.2 (Finite étale invariance of PEC)**.** *Let*

$$f\colon \mathcal{Y} \longrightarrow \mathcal{X}$$

be a representable finite étale surjective morphism of finite-type complex Deligne–Mumford stacks of constant degree d. Assume that $\mathcal{X}$ admits a representable finite étale atlas by a variety. Then

$$\mathcal{X} \text{ has PEC} \iff \mathcal{Y} \text{ has PEC.}$$

Proof. Suppose first that $\mathcal{Y}$ has PEC, and let $P \in \mathrm{Perv}(\mathcal{X}, \mathbb{Q})$. Since f is étale, f^*P is perverse. The finite étale degree formula [3, Theorem 2.5(iii)] gives

$$\chi^{\mathrm{orb}}(\mathcal{Y}, f^*P) = d\,\chi^{\mathrm{orb}}(\mathcal{X}, P).$$

The left-hand side is nonnegative, so $\chi^{\mathrm{orb}}(\mathcal{X}, P) \geq 0$.

Conversely, suppose that $\mathcal{X}$ has PEC, and let $Q \in \mathrm{Perv}(\mathcal{Y}, \mathbb{Q})$. We can choose a finite étale surjective atlas $u : U \to \mathcal{X}$ of degree m with U a variety. Consider the cartesian square

$$\begin{tikzcd}
V := U \times_{\mathcal{X}} \mathcal{Y} \arrow[r, "v"] \arrow[d, "f'"'] & \mathcal{Y} \arrow[d, "f"] \\
U \arrow[r, "u"'] & \mathcal{X}.
\end{tikzcd}$$

Since f is representable, V is an algebraic space, and since $f'\colon V \to U$ is finite, V is in fact a variety. Moreover, f' is finite étale of degree d, while v is finite étale of degree m, being the respective base changes of f and u. By the finite étale degree formula, proper base change, and the usual equality of Euler characteristics under finite pushforward, one has

$$\begin{aligned}
m\,\chi^{\mathrm{orb}}(\mathcal{Y}, Q) &= \chi(V, v^*Q) \\
&= \chi(U, Rf'_*v^*Q) \\
&= \chi(U, u^*Rf_*Q) \\
&= m\,\chi^{\mathrm{orb}}(\mathcal{X}, Rf_*Q).
\end{aligned}$$

Now $Rf_*Q \in \mathrm{Perv}(\mathcal{X}, \mathbb{Q})$. To see this, we need to show that u^*Rf_*Q is perverse on U a variety. Now f is finite, and thus proper. We can use proper base change which says $u^*Rf_*Q \simeq Rf'_*v^*Q$. Since v is étale v^*Q is perverse on V which is a variety and as f' is finite we get that $Rf'_*v^*Q \in \mathrm{Perv}(U, \mathbb{Q})$. Therefore

$$\chi^{\mathrm{orb}}(\mathcal{Y}, Q) = \chi^{\mathrm{orb}}(\mathcal{X}, Rf_*Q) \geq 0,$$

and $\mathcal{Y}$ has PEC. □

2.2. **The common-cover reduction theorem**

We reduce to a finite étale cover which is a product of neat locally symmetric varieties.

Proposition 2.3. *Let $\mathcal{D}$ be a Hermitian symmetric domain of noncompact type and let $\Gamma \subset G_{\mathcal{D}} = \mathrm{Aut}(\mathcal{D})^\circ$ be a lattice. There exist decompositions*

$$\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_r, \qquad \Lambda = \Lambda_1 \times \cdots \times \Lambda_r \subset \Gamma,$$

where Λ has finite index in Γ and each Λ_i is neat, such that, with

$$Y_i = \Lambda_i \backslash \mathcal{D}_i, \qquad Y = Y_1 \times \cdots \times Y_r,$$

the variety Y is smooth and quasi-projective and the natural map

$$Y \longrightarrow [\mathcal{D}/\Gamma]$$

is representable, finite étale, and surjective. Each Y_i is either an arithmetic locally symmetric variety or a finite-volume complex-ball quotient with unipotent parabolic elements.

Proof. The group $G_{\mathcal{D}}$ is adjoint semisimple and has no compact factors. By [29, Proposition 4.3.3], there is a decomposition $G_{\mathcal{D}} = G_1 \times \cdots \times G_r$ such that

$$\Gamma' = \Gamma_1 \times \cdots \times \Gamma_r \subset \Gamma, \qquad \Gamma_i = \Gamma \cap G_i,$$

has finite index and each Γ_i is an irreducible lattice. The corresponding decomposition of the symmetric space is $\mathcal{D} = \prod_i \mathcal{D}_i$.

If $\mathrm{rank}_{\mathbb{R}} G_i \geq 2$, then Γ_i is arithmetic by Margulis arithmeticity [23, Theorem 1]. Thus a nonarithmetic Γ_i has real rank one; since $\mathcal{D}_i$ is Hermitian, the classification of irreducible Hermitian symmetric spaces gives $\mathcal{D}_i \simeq \mathbb{B}^{n_i}$; see [15, Chapters VIII and X].

For an arithmetic factor choose a neat finite-index subgroup $\Lambda_i \subset \Gamma_i$ using [24, Proposition 3.5]; its quotient is smooth and quasi-projective by the Baily–Borel theorem [5]. For a nonarithmetic ball factor, choose a finite index neat subgroup $\Lambda_i \subset \Gamma_i$, such a subgroup exists by [10, Remark A.2]. Neatness implies that Λ_i is torsion-free and that all of its parabolic elements are unipotent; see [10, §A.1]. The quotient $\Lambda_i \backslash \mathbb{B}^{n_i}$ is smooth and quasi-projective by [25].

Hence $\Lambda = \Lambda_1 \times \cdots \times \Lambda_r$ is finite index and

$$Y = \Lambda \backslash \mathcal{D} = \prod_{i=1}^{r} \Lambda_i \backslash \mathcal{D}_i$$

is a smooth quasi-projective variety. Finally, the inclusion $\Lambda \subset \Gamma$ induces

$$[\mathcal{D}/\Lambda] \longrightarrow [\mathcal{D}/\Gamma].$$

After base change by the atlas $\mathcal{D} \to [\mathcal{D}/\Gamma]$, this is a disjoint union of $[\Gamma : \Lambda]$ copies of $\mathcal{D}$. It is therefore representable, finite étale, and surjective. Since Λ is neat,

$$[\mathcal{D}/\Lambda] \simeq Y,$$

which proves the proposition. □

Corollary 2.4 (PEC on the neat product cover)**.** *With the notation above,*

$$[\mathcal{D}/\Gamma] \text{ has PEC} \iff Y \text{ has PEC.}$$

Proof. The map

$$Y \longrightarrow [\mathcal{D}/\Gamma]$$

is finite étale and surjective, so the assertion follows from Proposition 2.2. □

2.3. Toroidal compactifications and products

We finish the section by recording the compactifications of the two building blocks and the product formula for their logarithmic cotangent bundles.

Proposition 2.5 (Compactifications of the building blocks)**.** *Let X_i be a factor appearing in proposition 2.3. Then X_i admits a smooth projective compactification*

$$X_i = \overline{X}_i \setminus D_i$$

for which D_i is a simple normal-crossings divisor. More precisely:

1. *if X_i is of arithmetic type (Shimura), one may choose a regular projective admissible rational polyhedral cone decomposition and take the associated toroidal compactification;*
2. *if X_i is of ball quotient type, one may take the smooth projective toroidal compactification obtained by filling in the cusps;*
3. *if X_i is compact, one takes $\overline{X}_i = X_i$ and $D_i = \varnothing$.*

Proof. For a Shimura factor, this is the toroidal compactification theorem of Ash-Mumford-Rapoport-Tai [4]. After refining an admissible rational polyhedral cone decomposition, it may be taken regular and projective. Regularity and neatness give smoothness and simple normal-crossings boundary, while projectivity of the decomposition gives projectivity of the compactification, see [4, Chapter III].

For a finite-volume complex-ball quotient whose parabolic elements are unipotent, the toroidal compactification is smooth and projective, with boundary a disjoint union of abelian varieties as in [25, Theorem 1]. The compact case corresponds to empty boundary. □

Lemma 2.6. *For $1 \le i \le r$, let $\overline{X}_i$ be a smooth projective variety and let $D_i \subset \overline{X}_i$ be a simple normal-crossings divisor. Put*

$$\overline{X} = \prod_{i=1}^{r} \overline{X}_i, \qquad D = \sum_{i=1}^{r} p_i^{-1}(D_i),$$

where $p_i \colon \overline{X} \to \overline{X}_i$ is the projection. Then $\overline{X}$ is smooth and projective, D is a simple normal-crossings divisor, and

$$\Omega^1_{\overline{X}}(\log D) \simeq \bigoplus_{i=1}^{r} p_i^* \Omega^1_{\overline{X}_i}(\log D_i). \tag{5}$$

In particular, if every $\Omega^1_{\overline{X}_i}(\log D_i)$ is nef, then $\Omega^1_{\overline{X}}(\log D)$ is nef.

Proof. One can check it locally, or [3, Lemma 2.3]. □

Corollary 2.7. *To prove Theorem 1.1, it is enough to establish logarithmic nefness for the following two classes:*

1. *connected components of Shimura varieties at neat finite level;*
2. *neat finite-volume complex-ball quotients with unipotent parabolic elements.*

More precisely, suppose that for every factor X_i in proposition2.3 one can choose the compactification of Proposition 2.5 so that

$$\Omega^1_{\overline{X}_i}(\log D_i)$$

is nef. Then $\mathcal{X}_\Gamma$ has the PEC property.

Proof. Let
$$X = X_1 \times \cdots \times X_r$$
be the product supplied by proposition 2.3. Choose smooth projective toroidal compactifications
$$X_i = \overline{X}_i \setminus D_i$$
with nef logarithmic cotangent bundles, and put
$$\overline{X} = \prod_{i=1}^{r} \overline{X}_i, \qquad D = \sum_{i=1}^{r} p_i^{-1}(D_i).$$
By Lemma 2.6,
$$\Omega^1_{\overline{X}}(\log D)$$
is nef. Theorem 2.1 therefore implies that X has PEC. By Corollary 2.4, $\mathcal{X}_\Gamma$ has PEC as well. □

Theorems 1.4 and 1.5 provide the two factorwise nefness statements in Corollary 2.7. Hence, after this section, no further stack-theoretic or group-theoretic reduction is needed. The remaining geometric verification takes place entirely on smooth neat quotients. So the boundary analysis reduces precisely to smooth noncompact connected components of finite-level Shimura varieties and smooth noncompact finite-volume complex-ball quotients.

3. Nef logarithmic cotangent bundle

3.1. **Arithmetic quotients**

Let $X = \Gamma\backslash\mathcal{D}$ be a connected component of a Shimura variety at neat finite level, and let
$$j\colon X \hookrightarrow \bar{X}$$
be a smooth projective toroidal compactification with
$$D = \bar{X} \setminus X$$
a simple normal-crossings divisor. Fix a point $h \in \mathcal{D}$, we can write $\mathcal{D} = G/K_h = \mathbf{G}(\mathbb{R})^+/K_h$ where $\mathbf{G}$ is an adjoint rational algebraic group. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be the Deligne torus, then equivalently, one can view $\mathcal{D}$ as a $\mathbf{G}(\mathbb{R})^+$-conjugacy class of a homomorphism $h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$. The adjoint complexified representation $\mathrm{Ad} \circ h : \mathbb{S}_{\mathbb{C}} \to \mathrm{Gl}(\mathfrak{g}_{\mathbb{C}})$ induces a decomposition
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_h^{-1,1} \oplus \mathfrak{g}_h^{0,0} \oplus \mathfrak{g}_h^{1,-1}$$
with the convention that $\mathrm{Ad}(h(z))$ acts by $z^{-p}\bar{z}^{p}$ on $\mathfrak{g}_h^{p,-p}$. We can define the Hodge filtration
$$F_h^1\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_h^{1,-1}, \quad F_h^0\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_h^{1,-1} \oplus \mathfrak{g}_h^{0,0}, \quad F_h^{-1}\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$$

For $x = g \cdot h \in \mathcal{D}$, the corresponding representation $h_x : \mathbb{S} \to \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ is obtained from h via conjugation by g. We then have the analogous decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_x^{-1,1} \oplus \mathfrak{g}_x^{0,0} \oplus \mathfrak{g}_x^{1,-1}$ and filtration $F_x^{\bullet}\mathfrak{g}_{\mathbb{C}}$.

3.1.1. *The adjoint polarized variation.* Let $P_h = \{g \in \mathbf{G}_{\mathbb{C}} | \ \mathrm{Ad}(g)F_h^p = F_h^p \text{ for all } p\}$ be the stabilizer of the filtration $F_h^{\bullet}$. Its Lie algebra is $\mathfrak{p}_h = F_h^0\mathfrak{g}_{\mathbb{C}}$, and we have the compact dual
$$\check{\mathcal{D}} = \mathbf{G}_{\mathbb{C}}/P_h$$
which is a flag variety. The Borel embedding
$$\beta : \mathcal{D} \to \check{\mathcal{D}}, \quad x \mapsto F_x^{\bullet}$$
is a $\mathbf{G}(\mathbb{R})^+$-equivariant holomorphic embedding which identifies $\mathcal{D}$ with an open orbit in $\check{\mathcal{D}}$.

Given W a finite-dimensional algebraic representation of P_h, we can define a homogeneous vector bundle on $\check{\mathcal{D}}$
$$\check{\mathcal{W}} = \mathbf{G}_{\mathbb{C}} \times^{P_h} W = (\mathbf{G}_{\mathbb{C}} \times W)/((g,w) \sim (gg', (g')^{-1}w)),$$

which is $\mathbf{G}_{\mathbb{C}}$-equivariant. Let

$$\check{\mathcal{V}} = \mathbf{G}_{\mathbb{C}} \times^{P_h} \mathfrak{g}_{\mathbb{C}}, \qquad \check{\mathcal{F}}^p = \mathbf{G}_{\mathbb{C}} \times^{P_h} F_h^p \mathfrak{g}_{\mathbb{C}}.$$

Since the adjoint representation of P_h on $\mathfrak{g}_{\mathbb{C}}$ extends to $\mathbf{G}_{\mathbb{C}}$, there is a $\mathbf{G}_{\mathbb{C}}$-equivariant isomorphism

$$\check{\mathcal{V}} \simeq \check{\mathcal{D}} \times \mathfrak{g}_{\mathbb{C}}, \qquad [g, v] \longmapsto (gP_h, \mathrm{Ad}(g)v).$$

Hence $\beta^* \check{\mathcal{V}} \simeq \mathcal{D} \times \mathfrak{g}_{\mathbb{C}}$, and under this identification the Γ-action is

$$\gamma \cdot (x, v) = (\gamma x, \mathrm{Ad}(\gamma)v).$$

Thus the adjoint representation $\mathrm{Ad}|_{\Gamma} : \Gamma \to \mathrm{GL}(\mathfrak{g}_{\mathbb{Q}})$ defines the rational local system

$$\mathbb{V}_{\mathbb{Q}} = \Gamma \backslash (\mathcal{D} \times \mathfrak{g}_{\mathbb{Q}}).$$

The bundles $\beta^* \check{\mathcal{V}}$ and $\beta^* \check{\mathcal{F}}^p$ are $\mathbf{G}(\mathbb{R})^+$-equivariant, hence Γ-equivariant, and therefore descend to an automorphic vector bundle $\mathcal{V}$ on X, together with a filtration by automorphic subbundles $\mathcal{F}^p \subset \mathcal{V}$. Moreover, $\mathcal{V} \simeq \mathbb{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_X$. If

$$\pi \colon \mathcal{D} \longrightarrow X = \Gamma \backslash \mathcal{D}$$

denotes the quotient map, then by construction

$$\pi^* \mathcal{V} \simeq \beta^* \check{\mathcal{V}}, \qquad \pi^* \mathcal{F}^p \simeq \beta^* \check{\mathcal{F}}^p.$$

Let ∇ be the flat connection on $\mathcal{V}$ obtained by descending the exterior derivative $1 \otimes \mathrm{d}$. Then

$$(\mathbb{V}_{\mathbb{Q}}, \mathcal{F}^{\bullet}, \nabla)$$

is a weight-zero rational variation of Hodge structures; see [24, Proposition 5.9].

Now let $\kappa(v, w) = \mathrm{Tr}(\mathrm{ad}(v)\,\mathrm{ad}(w))$ be the Killing form on $\mathfrak{g}_{\mathbb{Q}}$, and put $Q = -\kappa$. Then Q is rational, symmetric, nondegenerate, and $\mathrm{Ad}(\mathbf{G})$-invariant. Thus it descends to a flat bilinear form on $\mathbb{V}_{\mathbb{Q}}$.

The invariance of Q under $h_x(\mathbb{S})$ implies

$$Q\big(\mathfrak{g}_x^{p,-p}, \mathfrak{g}_x^{q,-q}\big) = 0 \qquad \text{unless } p + q = 0.$$

Indeed, if $v \in \mathfrak{g}_x^{p,-p}$ and $w \in \mathfrak{g}_x^{q,-q}$, then for every $z \in \mathbb{C}^{\times}$,

$$Q(v, w) = z^{-(p+q)} \overline{z}^{\,p+q} Q(v, w),$$

which forces $Q(v, w) = 0$ unless $p + q = 0$. Since Q is nondegenerate, it therefore gives perfect pairings

$$\mathfrak{g}_x^{p,-p} \times \mathfrak{g}_x^{-p,p} \longrightarrow \mathbb{C}.$$

Moreover, by the characterization of a Hermitian symmetric domain,

$$C_x = \mathrm{Ad}(h_x(i))$$

is a Cartan involution of $\mathfrak{g}_{\mathbb{R}}$. Hence the real symmetric form

$$(v, w) \longmapsto Q(v, C_x w) = -\kappa(v, C_x w)$$

is positive definite on $\mathfrak{g}_{\mathbb{R}}$. Equivalently, the associated Hodge Hermitian form is positive definite. Thus Q polarizes the weight-zero variation of Hodge structure $(\mathbb{V}_{\mathbb{Q}}, \mathcal{F}^{\bullet}, \nabla)$.

Now take the associated graded to get a system of Higgs bundles

$$(E, \theta) = \mathrm{gr}_{\mathcal{F}}(\mathcal{V}, \nabla), \quad \theta : E^{p,-p} \to E^{p-1,-p+1} \otimes \Omega_X^1$$

and $E^{-1,1} = \Gamma \setminus (\beta^*(\mathbf{G}_{\mathbb{C}} \times^{P_h} \mathrm{gr}_{F_h}^{-1} \mathfrak{g}_{\mathbb{C}}))$. At $h \in \mathcal{D} \subset \check{\mathcal{D}}$, the tangent space is

$$T_h \mathcal{D} = T_h \check{\mathcal{D}} \simeq \mathfrak{g}_{\mathbb{C}} / \mathfrak{p}_h = \mathfrak{g}_{\mathbb{C}} / F_h^0 \mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}_h^{-1,1}$$

and the same is true for any $x \in \mathcal{D}$. These tangent identifications are $\mathbf{G}(\mathbb{R})^+$-equivariant and holomorphically varying, hence $T_{\mathcal{D}}^{\mathrm{hol}} \simeq \beta^*(\mathbf{G}_{\mathbb{C}} \times^{P_h} \mathrm{gr}_{F_h}^{-1} \mathfrak{g}_{\mathbb{C}})$. Quotienting by Γ we get $T_X \simeq E^{-1,1}$. Also, $\theta(T_X) = \theta(E^{-1,1}) = 0$ since there is no $(-2, 2)$-graded piece.

3.1.2. *Deligne and Mumford canonical extensions.* We have an exact functor called canonical extension

$$(-)^{\mathrm{can}}\colon \{\text{automorphic vector bundles on } X\} \longrightarrow \{\text{vector bundles on } \bar{X}\};$$

see [26, §3, Main Theorem 3.1] and [13, §4]. The functor is compatible with tensor products and duals, and hence also with internal Hom. In particular, canonical extension preserves exact sequences of automorphic vector bundles.

The adjoint variation $(\mathbb{V}_{\mathbb{Q}}, \mathcal{F}^\bullet, \nabla, Q)$ has unipotent local monodromy around D [13, proof of Theorem 4.2], so we have Deligne's canonical extension [9] $(\bar{\mathcal{V}}, \bar{\nabla})$ which is a logarithmic flat bundle on $(\bar{X}, D)$, i.e., $\bar{V}$ is locally free over $\mathcal{O}_{\bar{X}}$ and

$$\bar{\nabla} : \bar{\mathcal{V}} \to \bar{\mathcal{V}} \otimes \Omega^1_{\bar{X}}(\log D)$$

By Harris's comparison theorem, the Mumford canonical extension of the flat automorphic bundle $\mathcal{V}$ agrees with Deligne's canonical extension; see [13, Theorem 4.2].

For each p, let

$$\mathcal{A}^p := (\mathcal{F}^p)^{\mathrm{can}}$$

denote the automorphic canonical extension of $\mathcal{F}^p$. Since

$$0 \longrightarrow \mathcal{F}^p \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{F}^p \longrightarrow 0$$

is an exact sequence of automorphic vector bundles, exactness of the canonical-extension functor gives

$$0 \longrightarrow \mathcal{A}^p \longrightarrow \bar{\mathcal{V}} \longrightarrow (\mathcal{V}/\mathcal{F}^p)^{\mathrm{can}} \longrightarrow 0.$$

In particular, the quotient is locally free, so $\mathcal{A}^p \subset \bar{\mathcal{V}}$ is saturated.

We claim that, inside $j_*\mathcal{V}$,

$$\mathcal{A}^p = (j_*\mathcal{F}^p) \cap \bar{\mathcal{V}}.$$

Indeed, the inclusion from left to right follows from $\mathcal{A}^p|_X = \mathcal{F}^p$. Conversely, a local section of $\bar{\mathcal{V}}$ whose restriction to X lies in $\mathcal{F}^p$ maps to a section of $\bar{\mathcal{V}}/\mathcal{A}^p$ which vanishes on the dense open subset X. Since this quotient is locally free, that section vanishes identically. Hence

$$(\mathcal{F}^p)^{\mathrm{can}} = (j_*\mathcal{F}^p) \cap \bar{\mathcal{V}} = \bar{\mathcal{F}}^p.$$

Thus the automorphic canonical extension of $\mathcal{F}^p$ agrees with the extended Hodge filtration inside Deligne's canonical extension $\bar{\mathcal{V}}$. In particular, $\bar{\mathcal{F}}^p \subset \bar{\mathcal{V}}$ is a locally free saturated subsheaf.

Applying exactness of the canonical-extension functor to

$$0 \longrightarrow \mathcal{F}^{p+1} \longrightarrow \mathcal{F}^p \longrightarrow E^{p,-p} \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow \bar{\mathcal{F}}^{p+1} \longrightarrow \bar{\mathcal{F}}^p \longrightarrow (E^{p,-p})^{\mathrm{can}} \longrightarrow 0.$$

Hence

$$\bar{\mathcal{F}}^p/\bar{\mathcal{F}}^{p+1} \simeq (E^{p,-p})^{\mathrm{can}}.$$

In particular, the graded pieces are locally free.

The logarithmic connection satisfies Griffiths transversality:

$$\bar{\nabla}(\bar{\mathcal{F}}^p) \subseteq \bar{\mathcal{F}}^{p-1} \otimes \Omega^1_{\bar{X}}(\log D).$$

Taking the associated graded therefore gives a logarithmic Higgs bundle

$$(\bar{E}, \bar{\theta}) = \mathrm{Gr}_{\bar{\mathcal{F}}}(\bar{\mathcal{V}}, \bar{\nabla}),$$

where

$$\bar{E}^{p,-p} = \bar{\mathcal{F}}^p/\bar{\mathcal{F}}^{p+1}, \qquad \bar{\theta}\colon \bar{E}^{p,-p} \longrightarrow \bar{E}^{p-1,-p+1} \otimes \Omega^1_{\bar{X}}(\log D).$$

Proposition 3.1. *There are canonical isomorphisms*

$$\bar{E}^{-1,1} \simeq T_{\overline{X}}(-\log D), \qquad \bar{F}^1 \simeq \Omega^1_{\overline{X}}(\log D) \simeq (\bar{E}^{-1,1})^\vee.$$

Proof. Recall that on X we have the canonical identification $E^{-1,1} \simeq T_X$. By the discussion above,

$$\bar{E}^{-1,1} = \mathrm{Gr}^{-1}_{\bar{\mathcal{F}}} \bar{\mathcal{V}} \simeq (E^{-1,1})^{\mathrm{can}} \simeq (T_X)^{\mathrm{can}}.$$

By [26, Proposition 3.4(a)],

$$(\Omega^1_X)^{\mathrm{can}} \simeq \Omega^1_{\bar{X}}(\log D).$$

Since canonical extension commutes with duals, it follows that

$$\bar{E}^{-1,1} \simeq (T_X)^{\mathrm{can}} \simeq T_{\bar{X}}(-\log D).$$

Dualizing gives $(\bar{E}^{-1,1})^\vee \simeq \Omega^1_{\bar{X}}(\log D)$. On X, the polarization induces an automorphic isomorphism $E^{1,-1} \simeq (E^{-1,1})^\vee$. Since canonical extension is compatible with duals, we obtain

$$\bar{E}^{1,-1} \simeq (\bar{E}^{-1,1})^\vee.$$

Finally, since $\mathcal{F}^2 = 0$, its canonical extension is also zero, so $\bar{\mathcal{F}}^2 = 0$. Therefore

$$\bar{\mathcal{F}}^1 = \bar{\mathcal{F}}^1/\bar{\mathcal{F}}^2 = \bar{E}^{1,-1}.$$

Combining these identifications yields

$$\bar{E}^{-1,1} \simeq T_{\bar{X}}(-\log D), \qquad \bar{\mathcal{F}}^1 \simeq \Omega^1_{\bar{X}}(\log D) \simeq (\bar{E}^{-1,1})^\vee.$$

□

Theorem 3.2 (Arithmetic nefness)**.** *Let $X = \Gamma\backslash\mathcal{D}$ be a quotient arising from a neat arithmetic subgroup of a rational group of Hermitian type. If $j \colon X \hookrightarrow \overline{X}$ is a smooth projective toroidal compactification whose reduced boundary $D = \overline{X} \setminus X$ has simple normal crossings, then*

$$\Omega^1_{\overline{X}}(\log D) \quad \textit{is nef.}$$

Proof. The bundle $\bar{E}^{-1,1}$ is a holomorphic subbundle of $\bar{E}$, and $\bar{\theta}(\bar{E}^{-1,1}) = 0$ since there is no $\bar{E}^{-2,2}$. Thus [7, Theorem 1.8] implies that $(\bar{E}^{-1,1})^\vee$ is nef. By Proposition 3.1,

$$(\bar{E}^{-1,1})^\vee \simeq \Omega^1_{\bar{X}}(\log D),$$

which proves the theorem. □

Remark 3.3. Alternatively, since $\mathcal{F}^1$ is the lowest nonzero piece of the Hodge filtration, we may apply [12, Corollary 1.2] to

$$\bar{\mathcal{F}}^1 \simeq \Omega^1_{\bar{X}}(\log D)$$

to obtain the desired nefness in this way as well.

3.2. **Ball quotients and completion of the argument**

The arithmetic ball quotient case follows from the previous argument, while the nonarithematic case is a direct application of the following proposition:

Proposition 3.4. [8, Theorem 3] *Let*

$$X = \Gamma\backslash\mathbb{B}^n,$$

where $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is a neat finite-covolume lattice whose parabolic elements are unipotent. Let

$$X \hookrightarrow \overline{X}$$

be its smooth projective toroidal compactification and let $D = \overline{X} \setminus X$ be the reduced boundary divisor. Then

$$\Omega^1_{\overline{X}}(\log D) \quad \textit{is nef.}$$

Thus all factors occurring in proposition 2.3 admit smooth projective logarithmic compactifications with nef logarithmic cotangent bundle. By 2.6 we get the nefness for the product as well.

4. Perverse Euler characteristics

In order to use nefness results to get our desired inequalities, we first recall some facts which are discussed in [3] and [2]. We will state it in the following generality.

Let $\overline{X}$ be a smooth projective variety of dimension n with $D \subset \overline{X}$ an SNC divisor. We denote $X = \overline{X} \setminus D$ for the quasi-projective variety. For a constructible complex K characteristic cycle $CC(K) \subset T^*X$ is a conical Lagrangian cycle of dimension n, see [17, Chapter IX] and [11, Sections 4.1–4.2]. If P is perverse, then $CC(P)$ is effective. Moreover, if P has generic rank r then

$$CC(P) = r[T^*_X X] + \sum_\alpha m_\alpha[\Lambda_\alpha], \qquad m_\alpha > 0,$$

where Λ_α are the remaining irreducible conical components. Let $\overline{CC(P)}$ denote the closure of $CC(P)$ in $\Omega^1_{\overline{X}}(\log D)$. The logarithmic Dubson-Kashiwara index formula of Wu-Zhou [30, Theorem 1.6] gives

$$\chi(X, P) = [\overline{CC(P)}] \cdot [s_0(\overline{X})]$$

where $s_0 : \overline{X} \to \Omega^1_{\overline{X}}(\log D)$ is the zero section.

If $\Omega^1_{\overline{X}}(\log D)$ is nef, positivity of conical cycles in a nef vector bundle [21, Example 8.2.6] shows that the intersection of each irreducible conical component with the zero section is nonnegative. This gives the criterion stated in 2.1 and gives $\chi(X, P) \geq 0$ in the nef log cotangent bundle case.

One can improve this result and get a sharper inequality as shown in [2].

Proposition 4.1 (Arapura–Mese–Patel [2])**.** *Let $\overline{X}$ be a smooth projective variety of dimension n with $D \subset \overline{X}$ an SNC divisor and $X := \overline{X} \setminus D$. Suppose that $\Omega^1_{\overline{X}}(\log D)$ is nef, let $P \in \mathrm{Perv}(X, \mathbb{Q})$ have generic rank r. Then*

$$\chi(X, P) \geq r(-1)^n \chi(X).$$

Consequently, if P has full support and $\chi(X) \neq 0$, then

$$\chi(X, P) > 0.$$

Hence, in the full support case, strict positivity reduces to showing that $\chi(X) \neq 0$. We have already established the required nefness statements for locally Hermitian symmetric spaces. We first deduce the corresponding nonnegativity results, and then verify that $\chi(X) \neq 0$ in the cases under consideration, which yields strict positivity.

4.1. Locally Hermitian symmetric varieties

Now we will prove the Euler characteristics result by combining the criterion given in 2.1 and the nefness results obtained above.

Theorem 4.2 (Locally Hermitian symmetric varieties)**.** *Let $\mathcal{D}$ be a Hermitian symmetric domain of noncompact type and let $\Gamma \subset \mathrm{Aut}(\mathcal{D})^\circ$ be a torsion-free lattice. Define $X_\Gamma = \Gamma \backslash \mathcal{D}$. Then X_Γ has the PEC property i.e.*

$$\chi(X_\Gamma, P) \geq 0 \qquad \text{for every } P \in \mathrm{Perv}(X_\Gamma, \mathbb{Q}).$$

In particular, if $n = \dim_{\mathbb{C}} X_\Gamma$, then

$$(-1)^n \chi(X_\Gamma) \geq 0.$$

Proof. By Proposition 2.3, there is a finite index neat subgroup $\Lambda = \Lambda_1 \times \cdots \times \Lambda_r \subset \Gamma$ such that

$$Y = \Lambda \backslash \mathcal{D} = Y_1 \times \cdots \times Y_r$$

is a smooth quasi-projective variety and

$$Y \longrightarrow X_\Gamma$$

is finite étale and surjective. Each Y_i is either an arithmetic locally symmetric variety or a finite-volume complex-ball quotient with unipotent parabolic elements.

Choose for each Y_i the smooth projective compactification $Y_i = \overline{Y}_i \setminus D_i$ of Proposition 2.5. By Theorem 3.2 and Proposition 3.4, respectively,

$$\Omega^1_{\overline{Y}_i}(\log D_i)$$

is nef for every i. Hence Lemma 2.6 shows that, for

$$\overline{Y} = \prod_i \overline{Y}_i, \qquad D = \overline{Y} \setminus Y,$$

the log cotangent bundle $\Omega^1_{\overline{Y}}(\log D)$ is nef. Theorem 2.1 therefore gives PEC for Y. Since $Y \to X_\Gamma$ is finite étale and surjective, Proposition 2.2 gives PEC for X_Γ. Finally, $\mathbb{Q}_{X_\Gamma}[n]$ is perverse because X_Γ is smooth of dimension n. Thus

$$0 \le \chi(X_\Gamma, \mathbb{Q}_{X_\Gamma}[n]) = (-1)^n \chi(X_\Gamma).$$

□

4.2. **Locally Hermitian symmetric Deligne–Mumford stacks**

We now recover the stack statement formally from the neat-cover reduction.

Theorem 4.3 (The stack case)**.** *Let $\mathcal{D}$ be a Hermitian symmetric domain of noncompact type and let $\Gamma \subset \mathrm{Aut}(\mathcal{D})^\circ$ be a lattice. Let $\mathcal{X}_\Gamma$ be the algebraic Deligne-Mumford stack constructed above, whose analytification is $\mathcal{X}_\Gamma^{\mathrm{an}} \simeq [\mathcal{D}/\Gamma]$. Then*

$$\chi^{\mathrm{orb}}(\mathcal{X}_\Gamma, P) \ge 0 \qquad \textit{for every } P \in \mathrm{Perv}(\mathcal{X}_\Gamma, \mathbb{Q}).$$

In particular, if $n = \dim_{\mathbb{C}} \mathcal{D}$, then

$$(-1)^n \chi^{\mathrm{orb}}(\mathcal{X}_\Gamma) \ge 0.$$

Proof. Let $Y = \Lambda \backslash \mathcal{D}$ be the smooth quasi-projective variety supplied by Proposition 2.3. By Theorem 4.2, Y has PEC. Moreover, Proposition 2.3 gives a representable finite étale surjective morphism of analytic Deligne–Mumford stacks

$$Y^{\mathrm{an}} \longrightarrow [\mathcal{D}/\Gamma] \simeq \mathcal{X}_\Gamma^{\mathrm{an}}.$$

Hence Proposition 2.2 implies that $\mathcal{X}_\Gamma$ has PEC.

Finally, $\mathcal{X}_\Gamma$ is smooth of dimension $n = \dim_{\mathbb{C}} \mathcal{D}$, so $\mathbb{Q}_{\mathcal{X}_\Gamma}[n]$ is perverse. Therefore

$$0 \le \chi^{\mathrm{orb}}(\mathcal{X}_\Gamma, \mathbb{Q}_{\mathcal{X}_\Gamma}[n]) = (-1)^n \chi^{\mathrm{orb}}(\mathcal{X}_\Gamma).$$

□

We also record the following corollary for finite level Shimura stacks.

Corollary 4.4 (Finite-level Shimura stacks)**.** *Let $(G, \mathcal{X})$ be a Shimura datum and let $K \subset G(\mathbb{A}_f)$ be an arbitrary compact open subgroup. Then*

$$\chi^{\mathrm{orb}}(\mathrm{Sh}_K^{\mathrm{DM}}(G, \mathcal{X})_{\mathbb{C}}, P) \ge 0$$

for every

$$P \in \mathrm{Perv}(\mathrm{Sh}_K^{\mathrm{DM}}(G, \mathcal{X})_{\mathbb{C}}, \mathbb{Q}).$$

If K is neat, then $\mathrm{Sh}_K^{\mathrm{DM}}(G, \mathcal{X})_{\mathbb{C}}$ is represented by the usual smooth finite-level Shimura variety, and χ^{orb} is the ordinary Euler characteristic.

Proof. Choose a normal neat compact open subgroup$K' \triangleleft K$. Then

$$\mathrm{Sh}_K^{\mathrm{DM}}(G, \mathcal{X}) = [\mathrm{Sh}_{K'}(G, \mathcal{X})/(K/K')],$$

so the natural morphism

$$\mathrm{Sh}_{K'}(G, \mathcal{X})_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K^{\mathrm{DM}}(G, \mathcal{X})_{\mathbb{C}}$$

is finite étale and surjective.

Since K' is neat, $\mathrm{Sh}_{K'}(G,\mathcal{X})_{\mathbb{C}}$ is a smooth quasi-projective variety, and its connected components are Hermitian locally symmetric varieties. Hence it has PEC by Theorem 4.2. Therefore Proposition 2.2 implies that $\mathrm{Sh}^{\mathrm{DM}}_K(G,\mathcal{X})_{\mathbb{C}}$ has PEC.

If K is neat, we may take $K'=K$, so $\mathrm{Sh}^{\mathrm{DM}}_K(G,\mathcal{X})_{\mathbb{C}}$ is the usual smooth Shimura variety and χ^{orb} is the ordinary Euler characteristic. □

4.3. **Positivity of Euler Characteristics**

To get strict inequality for $P\in\mathrm{Perv}(\mathcal{X}_\Gamma,\mathbb{Q})$ with full support, we first have a nonvanishing result:

Proposition 4.5. *Let $\mathcal{D}$ be a Hermitian symmetric domain of noncompact type, and $\Gamma\subset\mathrm{Aut}(\mathcal{D})^\circ$ be a torsion-free lattice. Put $X_\Gamma=\Gamma\setminus\mathcal{D}$. Then $\chi(X_\Gamma)\neq 0$.*

Proof. Euler characteristic is multiplicative, and its nonvanishing is preserved by finite covers, so by proposition 2.3, it suffices to assume X_Γ falls into one of the two cases: $\mathcal{D}$ is an irreducible Hermitian symmetric domain of dimension n with Γ an arithmetic neat lattice, or $\mathcal{D}=\mathbb{B}^n$ with Γ a neat lattice whose parabolic elements are unipotent. Let $(\overline{X}_\Gamma,D)$ be the toroidal compactification.

We claim that $\chi(X_\Gamma)\neq 0$ is equivalent to $\chi(\check{D})\neq 0$ where $\check{\mathcal{D}}$ is the compact dual of $\mathcal{D}$. In the arithmetic case, using the notations of section 3.1, we have $E^{-1,1}\simeq T_{X_\Gamma}$ on X_Γ with the corresponding vector bundle $\mathbf{G}_{\mathbb{C}}\times^{P_h}\mathrm{gr}^{-1}_{F_h}\mathfrak{g}_{\mathbb{C}}\simeq T_{\check{\mathcal{D}}}$ on the compact dual. We also have the Mumford canonical extension $\overline{E}^{-1,1}\simeq T_{\overline{X}_\Gamma}(-\log D)$. By [26, Theorem 3.2] we have

$$\int_{\overline{X}_\Gamma} c_n(T_{\overline{X}_\Gamma}(-\log D)) = (-1)^n\cdot K\cdot\int_{\check{\mathcal{D}}} c_n(T_{\check{\mathcal{D}}})$$

with K a positive constant. The same proportionality result is true in the nonarithmetic ball quotient case as well [16, §2]. Thus, by (logarithmic) Gauss-Bonnet [27, Theorem 3],

$$\chi(X_\Gamma)=(-1)^n\cdot K\cdot\chi(\check{\mathcal{D}})$$

Finally, $\chi(\check{\mathcal{D}})\neq 0$ since, by the Bruhat decomposition, $\check{\mathcal{D}}=\mathbf{G}_{\mathbb{C}}/P_h$ is a finite CW-complex with only even $\mathbb{R}$-dimensional cells. □

Corollary 4.6. *Let $\mathcal{D}$ be a Hermitian symmetric domain of noncompact type, and $\Gamma\subset\mathrm{Aut}(\mathcal{D})^\circ$ be a lattice. Denote $\mathcal{X}_\Gamma=[\Gamma\setminus\mathcal{D}]$ the Deligne-Mumford stack. If $P\in\mathrm{Perv}(\mathcal{X}_\Gamma,\mathbb{Q})$ has full support, then*

$$\chi^{\mathrm{orb}}(\mathcal{X}_\Gamma,P)>0$$

Proof. By [3, Theorem 2.5(iii)], we can assume that Γ is neat and it suffices to show that $\chi(X_\Gamma,P)>0$ for all perverse P with full support. This follows from proposition 4.1 and proposition 4.5. □

5. Applications

5.1. **Properties**

For applications we will need the following properties:

Proposition 5.1. *Let X be a complex algebraic variety having PEC. Then the following statements hold.*

(1) *Every closed subvariety*
$$i\colon Z\hookrightarrow X$$
has PEC.

(2) *Every locally closed subvariety*
$$j\colon U\hookrightarrow X$$
for which j is affine has PEC. In particular, this applies to the complement of an effective Cartier divisor.

(3) *Let U be either X or a subvariety occurring in* (1) *or* (2)*. If*

$$f\colon Y \longrightarrow U$$

is finite, then Y has PEC.

Proof. (1) Consider $P \in \mathrm{Perv}(Z, \mathbb{Q})$, then $Ri_*P = i_*P$ is perverse and $R\Gamma(X, i_*P) \simeq R\Gamma(Z, P)$. Hence $\chi(Z, P) = \chi(X, i_*P) \geq 0$.

(2) Let $P \in \mathrm{Perv}(U, \mathbb{Q})$. Since j is affine and quasi-finite, Rj_*P is perverse. Moreover, $R\Gamma(X, Rj_*P) \simeq R\Gamma(U, P)$. Therefore$\chi(U, P) = \chi(X, Rj_*P) \geq 0$.

(3) Let $P \in \mathrm{Perv}(Y, \mathbb{Q})$. Since f is finite, Rf_*P is perverse, and $R\Gamma(U, Rf_*P) \simeq R\Gamma(Y, P)$. Since U has PEC by (1) or (2), as appropriate, we obtain $\chi(Y, P) = \chi(U, Rf_*P) \geq 0$.

The required perverse t-exactness statements follow from [6, Corollaire 4.1.3]. □

One can upgrade this property to stacks but we do not need it in our examples. For moduli space applications we also record the following corollary.

Corollary 5.2 (Arrangement complements)**.** *Let $\mathcal{X} = [\mathcal{D}/\Gamma]$ be a finite-volume locally Hermitian symmetric Deligne–Mumford stack, and let $\mathcal{H} \subset \mathcal{D}$ be a Γ-invariant locally finite union of hypersurfaces. Suppose that there is a neat finite-index subgroup $\Lambda \subset \Gamma$ such that $X_\Lambda = \Lambda\backslash\mathcal{D}$ is a complex algebraic variety and the image $H_\Lambda = \Lambda\backslash\mathcal{H} \subset X_\Lambda$ is an algebraic effective Cartier divisor. Then*

$$[(\mathcal{D} \setminus \mathcal{H})/\Gamma]$$

has PEC.

Proof. By Theorem 4.2, the variety X_Λ has PEC. Since H_Λ is an effective Cartier divisor, the open immersion

$$X_\Lambda \setminus H_\Lambda \hookrightarrow X_\Lambda$$

is affine. Hence $X_\Lambda \setminus H_\Lambda$ has PEC by Proposition 5.1(2).

The natural morphism

$$X_\Lambda \setminus H_\Lambda \longrightarrow [(\mathcal{D} \setminus \mathcal{H})/\Gamma]$$

is representable, finite étale, and surjective. The conclusion follows from Proposition 2.2. □

5.2. **Moduli of smooth cubic threefolds**

Let $V = \mathrm{Sym}^3(\mathbb{C}^5)^\vee$, $G = \mathrm{PGL}_5$, and

$$\mathfrak{M}_0 = [\mathbb{P}(V)^{\mathrm{sm}}/G]$$

be the Deligne-Mumford stack of smooth cubic threefolds. Allcock–Carlson–Toledo associate to this moduli problem a complex ball $\mathbb{B}^{10}$, an arithmetic group $P\Gamma$, and two locally finite $P\Gamma$-invariant hyperplane arrangements $\mathcal{H}_c$ and $\mathcal{H}_\Delta$, called the chordal and discriminant arrangements. They construct the moduli space of smooth cubic threefolds as a complex analytic orbifold; see [1, §2]. Their period map identifies this orbifold with its image in the ball quotient, and [1, Theorem 7.1] identifies that image with the complement of the two arrangements. Thus there is an isomorphism of analytic Deligne–Mumford stacks

$$\mathfrak{M}_0^{\mathrm{an}} \simeq \left[(\mathbb{B}^{10} \setminus (\mathcal{H}_c \cup \mathcal{H}_\Delta))/P\Gamma\right].$$

Corollary 5.3. *The Deligne–Mumford stack $\mathfrak{M}_0$ has PEC, i.e.*

$$\chi^{\mathrm{orb}}(\mathfrak{M}_0, P) \geq 0 \qquad \textit{for every } P \in \mathrm{Perv}(\mathfrak{M}_0, \mathbb{Q}).$$

Moreover, if P has full support, then $\chi^{\mathrm{orb}}(\mathfrak{M}_0, P) > 0$. In particular, $\chi^{\mathrm{orb}}(\mathfrak{M}_0) > 0$.

Proof. Denote $\mathcal{H} = \mathcal{H}_c \cup \mathcal{H}_\Delta$. This is a $P\Gamma$-invariant locally finite hyperplane arrangement as shown in the proof of [1, Theorem 7.1] and [1, Lemma 7.2 and §8]. Choose a neat finite-index subgroup $\Lambda \subset P\Gamma$ and put $B_\Lambda = \Lambda\backslash\mathbb{B}^{10}$. The variety B_Λ is smooth and quasi-projective, and

$$H_\Lambda = \Lambda\backslash\mathcal{H} \subset B_\Lambda$$

is an algebraic divisor, being the pullback of the chordal and discriminant divisors appearing in [1, Theorem 7.1]. Since B_Λ is smooth, H_Λ is an effective Cartier divisor. Thus Corollary 5.2 gives PEC for

$$\left[(\mathbb{B}^{10} \setminus \mathcal{H})/P\Gamma\right],$$

and hence, by the period isomorphism, for $\mathfrak{M}_0$.

Now let P have full support and let P_Λ be its pullback to $U_\Lambda = B_\Lambda \setminus H_\Lambda$. Since $j\colon U_\Lambda \hookrightarrow B_\Lambda$ is affine, Rj_*P_Λ is perverse and has the same positive generic rank. Applying Proposition 4.1 to Rj_*P_Λ, together with $\chi(B_\Lambda) > 0$, gives

$$\chi(U_\Lambda, P_\Lambda) = \chi(B_\Lambda, Rj_*P_\Lambda) > 0.$$

Dividing by the degree of the finite étale cover $U_\Lambda \to \mathfrak{M}_0^{\mathrm{an}}$ yields

$$\chi^{\mathrm{orb}}(\mathfrak{M}_0, P) > 0.$$

Finally, $\mathbb{Q}_{\mathfrak{M}_0}[10]$ has full support, so $\chi^{\mathrm{orb}}(\mathfrak{M}_0) > 0$. □

5.3. **Moduli of smooth cubic fourfolds**

Let

$$V_4 = \mathrm{Sym}^3(\mathbb{C}^6)^\vee, \qquad G_4 = \mathrm{PGL}_6,$$

and let

$$\mathfrak{M}_4 = [\mathbb{P}(V_4)^{\mathrm{sm}}/G_4]$$

be the Deligne–Mumford stack of smooth cubic fourfolds. It is smooth of dimension 20.

Let Λ_0 be the primitive middle-cohomology lattice of a cubic fourfold, of signature $(20, 2)$, and let

$$\mathcal{D} \subset \mathbb{P}(\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{C})$$

be one of the two connected components of the associated type-IV period domain. The global monodromy group is

$$\Gamma = \mathrm{O}^*(\Lambda_0).$$

The arithmetic quotient $\mathcal{D}/\Gamma$ is quasi-projective; see [20, §2.1].

There are two Γ-invariant hyperplane arrangements $\mathcal{H}_6$ and $\mathcal{H}_2$. The first is determined by the short roots, i.e. the norm-2 vectors of Λ_0, while the second is determined by the long roots, i.e. the norm-6 vectors of divisibility 3. Their quotients

$$\mathcal{C}_6 = \mathcal{H}_6/\Gamma, \qquad \mathcal{C}_2 = \mathcal{H}_2/\Gamma$$

are Heegner divisors in $\mathcal{D}/\Gamma$; see [20, Definition 2.1 and Remark 2.2].

By [20, Theorem 2.3], which cites the global Torelli theorem of Voisin [28], the work of Hassett on special cubic fourfolds [14], and the determination of the period image by Laza [19, Theorem 1.1] and by Looijenga [22], the period map induces an isomorphism of quasi-projective varieties

$$\mathcal{M}_4 \simeq \Gamma\backslash(\mathcal{D} \setminus (\mathcal{H}_2 \cup \mathcal{H}_6)).$$

As stated immediately after [20, Theorem 2.3], identifying the natural orbifold structures on the two sides is equivalent to the strong global Torelli theorem. This is supplied by [20, Proposition 2.4]. Hence the period map upgrades to an isomorphism of analytic Deligne–Mumford stacks

$$\mathfrak{M}_4^{\mathrm{an}} \simeq \left[(\mathcal{D} \setminus (\mathcal{H}_2 \cup \mathcal{H}_6))/\Gamma\right]. \tag{6}$$

Corollary 5.4. *The Deligne–Mumford stack $\mathfrak{M}_4$ has PEC. Thus*

$$\chi^{\mathrm{orb}}(\mathfrak{M}_4, P) \geq 0 \qquad \text{for every } P \in \mathrm{Perv}(\mathfrak{M}_4, \mathbb{Q}).$$

Moreover, if P has full support, then $\chi^{\mathrm{orb}}(\mathfrak{M}_4, P) > 0$. In particular, $\chi^{\mathrm{orb}}(\mathfrak{M}_4) > 0$.

Proof. Set

$$\mathcal{H}_4 = \mathcal{H}_2 \cup \mathcal{H}_6.$$

By [20, Definition 2.1], this is a Γ-invariant arithmetic hyperplane arrangement, and

$$\mathcal{C}_2 = \mathcal{H}_2/\Gamma, \qquad \mathcal{C}_6 = \mathcal{H}_6/\Gamma$$

are algebraic divisors on the quasi-projective variety $\mathcal{D}/\Gamma$; see also [20, Remark 2.2].

Choose a neat finite-index subgroup $\Lambda \subset \Gamma$, and put $B_\Lambda = \Lambda \backslash \mathcal{D}$. By the Baily–Borel theorem, B_Λ is a smooth quasi-projective variety, and the natural map

$$\pi_\Lambda \colon B_\Lambda \longrightarrow \mathcal{D}/\Gamma$$

is finite and algebraic. Let $H_{4,\Lambda} = \pi_\Lambda^{-1}(\mathcal{C}_2 \cup \mathcal{C}_6)$. Then $H_{4,\Lambda} \subset B_\Lambda$ is an algebraic divisor. Since B_Λ is smooth, $H_{4,\Lambda}$ is an effective Cartier divisor. Hence

$$B_\Lambda \setminus H_{4,\Lambda} \hookrightarrow B_\Lambda$$

is an affine open immersion. Thus the hypotheses of Corollary 5.2 are satisfied, and

$$\big[(\mathcal{D} \setminus \mathcal{H}_4)/\Gamma\big]$$

has PEC. By the period isomorphism (6), $\mathfrak{M}_4$ has PEC.

Moreover, if $P \in \mathrm{Perv}(\mathfrak{M}_4, \mathbb{Q})$ has full support, then its pullback to $U_\Lambda = B_\Lambda \setminus H_{4,\Lambda}$ has full support and positive generic rank. Since $U_\Lambda \hookrightarrow B_\Lambda$ is affine, its direct image is perverse with the same generic rank. Proposition 4.1, together with $\chi(B_\Lambda) > 0$, therefore gives $\chi(U_\Lambda, P_\Lambda) > 0$. Dividing by the degree of the finite étale cover $U_\Lambda \to \mathfrak{M}_4^{\mathrm{an}}$ yields

$$\chi^{\mathrm{orb}}(\mathfrak{M}_4, P) > 0.$$

□

Acknowledgments

The authors thank Donu Arapura for helpful discussions and suggestions.

References

1. Daniel Allcock, James A. Carlson, and Domingo Toledo, *The moduli space of cubic threefolds as a ball quotient*, Mem. Amer. Math. Soc. **209** (2011), no. 985, xii+70.
2. Donu Arapura, Chikako Mese, and Deepam Patel, *A higher dimensional log riemann–hurwitz inequality and rigidity of covers*, Unpublished manuscript, 2026.
3. Donu Arapura and Deepam Patel, *Nonnegativity of signed euler characteristics of moduli of curves and abelian varieties*, Math. Ann. **394** (2026), Article No. 34.
4. Armand Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties*, 2 ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, With the collaboration of Peter Scholze.
5. Jr. Baily, Walter L. and Armand Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.
6. Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, Analysis and Topology on Singular Spaces, I (Luminy, 1981), Astérisque, vol. 100, Société Mathématique de France, Paris, 1982, pp. 5–171.
7. Yohan Brunebarbe, *Symmetric differentials and variations of Hodge structures*, J. Reine Angew. Math. **743** (2018), 133–161.
8. Benoît Cadorel, *Symmetric differentials on complex hyperbolic manifolds with cusps*, J. Differential Geom. **118** (2021), no. 3, 373–398.
9. Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, vol. 163, Springer-Verlag, Berlin–New York, 1970.

10. Ya Deng, *A characterization of complex quasi-projective manifolds uniformized by unit balls*, Math. Ann. **384** (2022), 1833–1881, With an appendix by Ya Deng and Benoît Cadorel.
11. Alexandru Dimca, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004.
12. Osamu Fujino and Taro Fujisawa, *On semipositivity theorems*, Math. Res. Lett. **26** (2019), no. 5, 1359–1382.
13. Michael Harris, *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc. (3) **59** (1989), no. 1, 1–22.
14. Brendan Hassett, *Special cubic fourfolds*, Compos. Math. **120** (2000), no. 1, 1–23.
15. Sigurdur Helgason, *Differential geometry, lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.
16. Jun-Muk Hwang, *On the volumes of complex hyperbolic manifolds with cusps*, Internat. J. Math. **15** (2004), no. 6, 567–572.
17. Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, 1990.
18. Yves Laszlo and Martin Olsson, *Perverse t-structure on Artin stacks*, Math. Z. **261** (2009), no. 4, 737–748.
19. Radu Laza, *The moduli space of cubic fourfolds via the period map*, Ann. of Math. (2) **172** (2010), no. 1, 673–711.
20. Radu Laza and Zhiwei Zheng, *Automorphisms and periods of cubic fourfolds*, Math. Z. **300** (2022), no. 2, 1455–1507.
21. Robert Lazarsfeld, *Positivity in algebraic geometry ii: Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 49, Springer-Verlag, Berlin, 2004.
22. Eduard Looijenga, *The period map for cubic fourfolds*, Invent. Math. **177** (2009), no. 1, 213–233.
23. G. A. Margulis, *Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than one*, Invent. Math. **76** (1984), no. 1, 93–120.
24. J. S. Milne, *Introduction to Shimura varieties*, Harmonic Analysis, the Trace Formula, and Shimura Varieties, Clay Mathematics Proceedings, vol. 4, American Mathematical Society, Providence, RI, 2005, pp. 265–378.
25. Ngaiming Mok, *Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume*, Perspectives in Analysis, Geometry, and Topology, Progress in Mathematics, vol. 296, Birkhäuser/Springer, New York, 2012, pp. 331–354.
26. David Mumford, *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239–272.
27. Yoshiki Norimatsu, *Kodaira vanishing theorem and chern classes for ∂-manifolds*, Proc. Japan Acad. Ser. A Math. Sci. **54** (1978), no. 4, 107–108.
28. Claire Voisin, *Théorème de Torelli pour les cubiques de* $\mathbf{P}^5$, Invent. Math. **86** (1986), no. 3, 577–601.
29. Dave Witte Morris, *Introduction to arithmetic groups*, Deductive Press, 2015.
30. Lei Wu and Peng Zhou, *Log $\mathscr{D}$-modules and index theorems*, Forum Math. Sigma **9** (2021), e3, 32 pp.

Department of Mathematics, Purdue University, West Lafayette, Indiana, USA
Email address: `makdeniz@purdue.edu`

Department of Mathematics, University of Chicago, Chicago, Illinois, USA
Email address: `quocanh@uchicago.edu`